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LETTER TO THE EDITOR

Conformal invariance and linear defects in the two-dimensional Ising model

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Abstract. Using conformal invariance, we show that the non-universal exponent η_0 associated with the decay of correlations along a defect line of modified bonds in the square-lattice Ising model is related to the amplitude $A_0 = \xi_n/n$ of the correlation length $\xi_n(K_c)$ at the bulk critical coupling K_c, on a strip with width n, periodic boundary conditions and two equidistant defect lines along the strip through $A_0 = (\pi\eta_0)^{-1}$.

In addition to scale invariance, a statistical system at its critical point also displays conformal invariance (Polyakov 1970, Fisher 1973, Wegner 1976). This property may be used to constrain the form of the correlation functions. In two-dimensional systems it even completely determines the critical exponents and correlation functions in the bulk (Belavin *et al* 1984, Dotsenko 1984, Friedan *et al* 1984) or near a surface (Cardy 1984a).

Recently conformal invariance has also been used (Cardy 1984b) to justify a remarkable universal relation between the amplitude A_0 of the correlation length ξ_n on a strip with periodic boundary conditions and the bulk critical exponent η for large n values:

$$\xi_n = A_0 n \tag{1}$$

$$A_0 = (\pi\eta)^{-1}. \tag{2}$$

This relation was first established in the case of Anderson localisation (Pichard and Sarma 1981); it was verified on the 2D XY model (Luck 1982), the Potts model (Derrida and de Sèze 1982) and generalised for correlations other than of the order-order type and anisotropic systems (Nightingale and Blöte 1983) and for quantum systems (Penson and Kolb 1984).

The purpose of the present letter is to establish a similar relation between the correlation length amplitude A_0 in the strip geometry and the non-universal exponent η_0 which characterises the decay of the order parameter correlation function at the critical point near a defect line in the 2D Ising model through a conformal transformation.

We shall consider two types of defect lines, the ladder and chain cases where the interaction strength K is modified along a straight line on bonds which are either perpendicular or parallel to the line (figure 1). Exact results have been obtained

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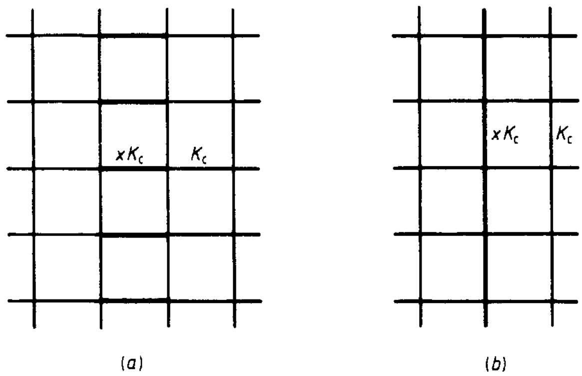

Figure 1. Defect line in: (a) the ladder geometry and (b) the chain geometry along which the interaction strength is modified ($K' = xK$, where K is the bulk interaction strength).

showing that for the Ising model the defect exponent η_0 is non-universal (Bariev 1980, McCoy and Perk 1980). For two points at a fixed distance from the defect, the correlation function at the critical point of the bulk decays like

$$\langle \varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\rangle \sim r^{-\eta_0} \qquad (r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|) \tag{3}$$

where $\varphi(r)$ is the Ising scalar field. The dependence on the defect strength for the square lattice is as follows

$$\eta_0 = \left(\frac{1}{\pi}\cos^{-1}(\chi)\right)^2 \tag{4}$$

where

$$\chi = \frac{\cosh(2xK_c) - \cosh(2K_c)}{\cosh(2K_c)\cosh(2xK_c) - 1} \tag{5}$$

in the ladder geometry and

$$\chi = \tanh[2K_c(x-1)] \tag{6}$$

in the chain geometry. $K_c = \frac{1}{2}\ln(1+\sqrt{2})$ is the Ising square lattice critical coupling.

Following Cardy (1984b) we use a finite conformal tranformation

$$w(z) = \ln(z) \tag{7}$$

to map the plane with a linear defect into a strip $|\mathrm{Im}(w)| \leq \pi$ with *two linear defects* along the strip at $|\mathrm{Im}(w)| = \pi/2$ and periodic boundary conditions (figure 2). Conformal covariance of the correlation function leads to

$$\langle \varphi(z_1)\varphi(z_2)\rangle = |w'(z_1)|^x |w'(z_2)|^x \langle \varphi(w_1)\varphi(w_2)\rangle \tag{8}$$

where $x = \eta/2$ is the anomalous dimension of the field φ. Writing $z_j = \exp(t_j + \mathrm{i}\theta_j)$, we get

$$\langle \varphi(t_1 + \mathrm{i}\theta_1)\varphi(t_2 + \mathrm{i}\theta_2)\rangle_s = \exp[x(t_1 + t_2)]\langle \varphi(z_1)\varphi(z_2)\rangle \tag{9}$$

where the correlation function on the left-hand side is evaluated in the strip geometry whereas on the right-hand side it is taken on the infinite plane. Through an infinitesimal conformal transformation, one may show that in the infinite plane with a defect line,

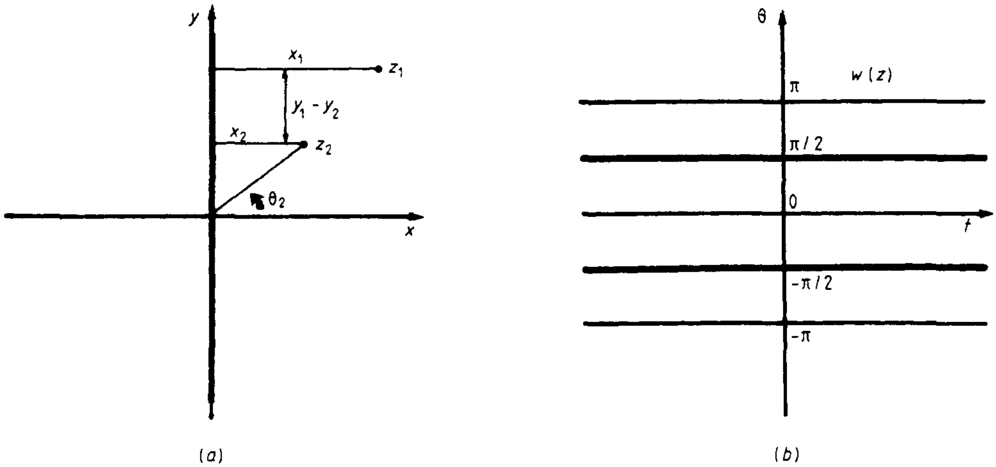

Figure 2. The conformal transformation $w = \ln(z)$ maps the plane with a defect line (a) into a strip with width 2π and two defect lines (b).

like in the case of a surface (Cardy 1984a), the correlation function behaves as follows

$$\langle\varphi(z_1)\varphi(z_2)\rangle = |x_1x_2|^{-x}\Phi\left(\frac{(y_1-y_2)^2+x_1^2+x_2^2}{x_1x_2}\right) \tag{10}$$

where $z_j = x_j + \mathrm{i}y_j$. The asymptotic behaviour near the defect requires that $\Phi(\rho)$ behaves as ρ^{-x_0} where x_0 is a defect exponent $(x_0 = \eta_0/2)$ as its argument $\rho \to \infty$. Changing for the strip variables, one gets

$$\langle\varphi(z_1)\varphi(z_2)\rangle = \exp[-x(t_1+t_2)]|\cos\theta_1\cos\theta_2|^{-x}$$
$$\times\Phi\left(\frac{\exp(t_1-t_2)+\exp(t_2-t_1)-2\sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2}\right) \tag{11}$$

and in the limit where $t_1 - t_2 \to +\infty$

$$\langle\varphi(z_1)\varphi(z_2)\rangle \sim \exp[-x(t_1+t_2)-x_0(t_1-t_2)]|\cos\theta_1\cos\theta_2|^{x_0-x} \tag{12}$$

so that on the strip

$$\langle\varphi(t_1+\mathrm{i}\theta_1)\varphi(t_2+\mathrm{i}\theta_2)\rangle_\mathrm{s} \sim f(\theta_1,\theta_2)\exp[-x_0(t_1-t_2)] \tag{13}$$

and the correlations along the strip with two defect lines decrease exponentially with a correlation length

$$\xi = 1/x_0 = 2/\eta_0 \tag{14}$$

measured in units where the width of the strip is 2π. If one introduces a lattice, n lattice spacings wide, the correlation length ξ_n, measured in lattice spacing units, satisfies:

$$\xi_n\frac{2\pi}{n} = \frac{2}{\eta_0} \tag{15}$$

$$\xi_n = (\pi\eta_0)^{-1}n \tag{16}$$

so that the universal relation between the correlation length amplitude and the decay exponent remains true if one replaces the bulk exponent η by the defect exponent η_0 and works on a strip with two defect lines.

The universal relation has been tested numerically using the transfer matrix method on strips with increasing size ($n=4, 6, 8$) and varying the defect strength ($x=0$ to 2) either in the ladder or in the chain geometry. The results are given in figure 3. The agreement with equation (16) is satisfactory although the convergence towards the exact curve on the side where the defect interaction is weaker than in the bulk remains rather slow. In order to test this convergence, we have extended the calculations up

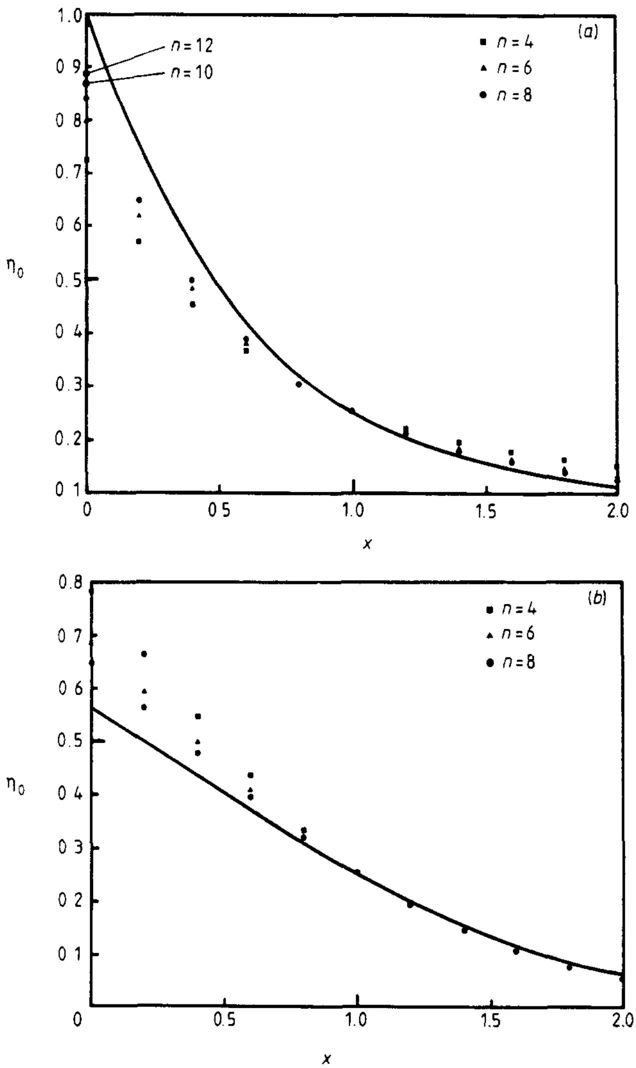

Figure 3. Defect exponent η_0 as a function of $x = K'/K_c$ where K' is the modified interaction along the defect line in the ladder (a) and chain (b) geometries. The full curve gives the exact results. Approximate results are deduced from the correlation length amplitude $A_0 = \xi_n/n = (\pi\eta_0)^{-1}$ on strips with width $n=4$, 6, 8. In the ladder geometry results are given for n up to 12 when $x=0$. The convergence is slow on the side where the interaction strength is weaker than in the bulk.

to $n = 12$ in the limit $x = 0$ with the ladder geometry. Then the system decouples into two strips with free boundary conditions and with half the width of the initial strip with periodic boundary conditions so that one may work on narrower strips to get equivalent results. Clearly the convergence is quite slow there.

To conclude let us mention that various defect geometries may be studied using the strip method. For instance a half-infinite defect line leads to a strip with a single defect line and periodic boundary conditions whereas a defect line perpendicular to a free surface corresponds to a strip with a single defect line and free boundary conditions.

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